Stable bundles on regular elliptic surfaces

Lübke, M ; Okonek, C

DOI: https://doi.org/10.1515/crll.1987.378.32

Originally published at:
Lübke, M; Okonek, C (1987). Stable bundles on regular elliptic surfaces. Journal für die Reine und Angewandte Mathematik, 378:32-45.
DOI: https://doi.org/10.1515/crll.1987.378.32

J. reine angew. Math. 378 (1987), 32—45

Journal für die reine und angewandte Mathematik

Stable bundles on regular elliptic surfaces

By *M. Lübke**) at Leiden and *C. Okonek***) at Göttingen

0. Introduction

In this paper we study stable and simple vector bundles on simply connected elliptic surfaces.

Let $\pi: X \longrightarrow \mathbb{P}_1$ be such a surface with geometric genus p_g. If $p_g = 0$, then X is a projective algebraic surface. Moduli spaces of stable rank-2 bundles on these surfaces have been computed in [21]. Here we extend the techniques from [21] and consider elliptic surfaces of higher geometric genus. Now the surfaces are in general non-algebraic, so we have to be careful with what we mean by stability.

Since every simply connected elliptic surface has a Kähler structure [17], it makes sense to consider ω-stable bundles for a suitable Kähler form ω [9]. If ω is induced by an ample line bundle L, this notion coincides with L-stability in the sense of Mumford-Takemoto.

Furthermore, from the proof of the conjecture of Kobayashi and Hitchin by Uhlenbeck and Yau [23] (see also [4]) we get a one to one correspondence between isomorphism classes of ω-stable bundles and gauge equivalence classes of irreducible projectively anti-self-dual unitary connections (i.e. Yang-Mills connections, i.e. Hermitian-Einstein bundles), so we can use results from Yang-Mills theory.

We adress the following questions:

1) What are the possible second Chern classes of ω-stable and simple rank-2 bundles with trivial determinant on a given surface X?

2) Can one describe the corresponding moduli spaces?

The expected dimension of the moduli space of stable 2-bundles with Chern classes $c_1 = 0$, c_2 is $4c_2 - 3(p_g + 1)$, so one expects such bundles for all $c_2 \geqq \left[\frac{3}{4}(p_g + 1)\right]$. In [22] Taubes proved a general existence theorem for anti-self-dual $SU(2)$-connections over arbitrary Riemannian 4-manifolds, which in our case for generic ω implies

*) Partially supported by the SFB 170 in Göttingen.

**) Supported by the Heisenberg-program of the DFG.

existence of ω-stable bundles for $c_2 \geqq 2p_g + 1$. We show, that for a good choice of the Kähler form there exist ω-stable 2-bundles with $c_1 = 0$, $c_2 = 1$ on all simply connected elliptic surfaces which have at least 2 multiple fibres with multiplicities greater than 1, thus showing that Taubes bound in [22] is in general not best possible. In fact, this was one of our motivations for this work.

If the surface is algebraic and ω is induced by a polarization L, we compute the moduli spaces of ω-stable 2-bundles with $c_1 = 0$, $c_2 = 1$. These spaces turn out to be finite disjoint unions of in general nowhere reduced curves. If the surface is non-algebraic, we get a complex structure in the moduli space of ω-stable bundles in the following way: The moduli space of simple bundles is a (not necessarily Hausdorff) complex space [19], and the ω-stable bundles form a (globally Hausdorff) open subset [14]. We determine the space of simple bundles and see that the (in general proper) subset of ω-stable ones admits the same description as in the algebraic case.

This work was supported by the Sonderforschungsbereich 170 in Göttingen; the authors thank it for the hospitality.

The second author takes this opportunity to thank A. van de Ven who introduced him to the subject.

1. Rank-2 vector bundles on elliptic Kähler surfaces

The bundles we shall study will appear as certain extensions, so let us first recall the following general fact from [21]:

Proposition 1. 1. *Let X be a compact complex surface with canonical bundle K, L any line bundle on X and $Z \subset X$ a finite set of simple points. Then there exists an extension*

$$(*) \qquad 0 \to L \otimes K^{-1} \to E \to J_Z \otimes L^{-1} \otimes K \to 0$$

with E locally free of rank 2, if (and only if in case $Z = \{z\}$ is a single point) Z is contained in the zero-set of every section of $(K^{\otimes 3}) \otimes (L^{-1})^{\otimes 2}$. For given $Z = \{z\}$ and L, the bundle E is uniquely determined by the existence of such an extension as soon as $h^1(L^{\otimes 2} \otimes (K^{-1})^{\otimes 2}) = 0$.

To characterize simple bundles we use

Proposition 1. 2. *Let X, Z be as above, L a line bundle on X with $h^0(J_Z \otimes (L^{-1})^{\otimes 2} \otimes K^{\otimes 2}) = 0$, and let E be a bundle defined by an extension $(*)$. Then E is simple if and only if $h^0(L^{\otimes 2} \otimes (K^{-1})^{\otimes 2}) = 0$.*

Proof. From $(*)$ one sees $c_1(E) = 0$ and therefore $E^* = E$. Thus by tensoring with E we get the exact sequence

$$0 \to E \otimes L \otimes K^{-1} \to \operatorname{End} E \to J_Z \otimes E \otimes L^{-1} \otimes K \to 0$$

with $H^0(\operatorname{End} E) \to H^0(J_Z \otimes E \otimes L^{-1} \otimes K)$ non-zero.

Since the last vector space is 1-dimensional, E is simple if and only if

$$h^0(L^{\otimes 2} \otimes (K^{-1})^{\otimes 2}) = h^0(E \otimes L \otimes K^{-1}) = 0.$$

Now let X be a compact Kähler surface with Kähler form ω. As in [9], [13] we define ω-(semi-)stability as follows: Let F be a coherent torsion-free sheaf of rank r on X; then $\det F = (\Lambda^r F)^{**}$ is a line bundle. Choose a hermitian metric h in $\det F$ and consider the first Chern form $c_1(\det F, h)$ associated to the metric connection. Now define the ω-degree of F by

$$\deg_\omega(F) = \int_X c_1(\det F, h) \wedge \omega.$$

Note that this number depends on the Kähler class of ω, but not on the metric in $\det F$; we often omit the subscript ω if it is clear which Kähler class we use.

Definition. F is called ω-(semi-)stable if for every rank-s subsheaf $F' \subset F$ with $0 < s < r$ one has

$$\frac{\deg(F')}{s} < \frac{\deg(F)}{r} \quad \left(\frac{\deg(F')}{s} \leqq \frac{\deg(F)}{r}\right).$$

If ω is induced by an ample line bundle L, then ω-stability is the same as L-stability. Of course every ω-stable bundle is simple. We use the following criterion for ω-(semi-)stability:

Proposition 1. 3. *Let L, Z be as before with* $\deg(L \otimes K^{-1}) < 0 \quad (\leqq 0)$,

$$h^0(J_Z \otimes (L^{-1})^{\otimes 2} \otimes K^{\otimes 2}) = 0,$$

and let E be a bundle defined by extension $(*)$. *Then E is ω-(semi-)stable if and only if for every curve R containing Z with $h^0(E \otimes L \otimes K^{-1} \otimes \mathcal{O}(R)) \neq 0$ one has*

$$\deg(L \otimes K^{-1} \otimes \mathcal{O}(R)) > 0 \quad (\geqq 0).$$

Proof. From $(*)$ we get $\deg(E) = 0$, so E is not stable if and only if there exists a monomorphism $L_1 \hookrightarrow E$, where L_1 is a line bundle with $\deg(L_1) \geqq 0$. Since $\deg(L \otimes K^{-1}) < 0$, this can only happen if there is a curve R containing Z with $\mathcal{O}(R) = L^{-1} \otimes K \otimes L_1^{-1}$, hence

$$\deg(L \otimes K^{-1} \otimes \mathcal{O}(R)) = \deg(L_1^{-1}) \leqq 0.$$

The converse is obvious, as is the semistable case.

From now on, let $X \longrightarrow \mathbb{P}_1$ be an elliptic surface, and denote by F_{p_i}, $i=1,\ldots,m$, the multiple fibres of X with multiplicities $2 \leqq p = p_1 \leqq \cdots \leqq p_m = q$.

We assume that all fibres are irreducible, $m \geqq 2$, that X has vanishing irregularity $q(X)=0$, and that X admits a Kähler form ω_0 (the surfaces in question later all have these properties).

Now the canonical divisor of X is given by

$$K \sim (p_g - 1)F + \sum_{i=1}^{m} (p_i - 1) F_{p_i}$$

where $p_g = p_g(X)$ and F is a generic fibre; in particular we have $K^2 = 0$ [2].

To get a good hold on stable bundles we modify ω_0 in the following way: If ω_1 is the Kähler form of the Fubini-Study-metric on $\mathbb{P}_1$, define

$$\omega = \omega_0 + n\pi^*(\omega_1)$$

with a fixed number $n > \max(\deg_{\omega_0}(K), 1)$, where $\pi\colon X \longrightarrow \mathbb{P}_1$ is the projection. Then ω again is a Kähler form, and in this setting we get

Corollary 1.4. *If the line bundle L in Proposition* 1.3 *satisfies* $\deg_{\omega_0}(L) \geqq 0$ *and* $\int_X c_1(L) \wedge \pi^*(\omega_1) = 0$ *(in particular if L is the line bundle associated to a curve C which is vertical, i.e. $C \cdot F = 0$), then a curve R violating the ω-(semi-)stability of E is vertical.*

If

$$L = \mathcal{O}(C),\quad C = aF + \sum_{i=1}^{m} x_i F_{p_i} \quad \text{and} \quad R = bF + \sum_{i=1}^{m} y_i F_{p_i}$$

with $0 \leqq a, b$ and $0 \leqq x_i,\ y_i < p_i$, then the condition $\deg_\omega(L \otimes K^{-1} \otimes \mathcal{O}(R)) > 0$ $(\geqq 0)$ *reads*

$$a + b + \sum_{i=1}^{m} \frac{x_i + y_i + 1}{m} - p_g - m + 1 > 0 \quad (\geqq 0).$$

Proof. First note, that $\pi^*(\omega_1)$ is a first Chern form of the line bundle $\mathcal{O}(F)$, so for every vertical curve C

$$\int_X c_1(\mathcal{O}(C)) \wedge \pi^*(\omega_1) = C \cdot F = 0.$$

Therefore a necessary condition for R to violate stability is

$$\begin{aligned} 0 &\geqq \deg_\omega(L \otimes K^{-1} \otimes \mathcal{O}(R)) \\ &= \deg_{\omega_0}(L \otimes K^{-1} \otimes \mathcal{O}(R)) + n \int_X c_1(R) \wedge \pi^*(\omega_1) \\ &\geqq -\deg_{\omega_0}(K) + nR \cdot F. \end{aligned}$$

Now R is a curve, so $0 \leqq R \cdot F \in \mathbb{Z}$, and the choice of n gives $R \cdot F = 0$.

We are now in the position to prove existence theorems.

Theorem 1.5. *If $3 \leqq p$ $(3 \leqq p < q)$, then for all $c_2 \geqq 1$ there is a simple 2-bundle on X with Chern classes $c_1 = 0$, c_2, which is not ω-(semi-)stable.*

Remark. For $p = 2$ we later shall see examples of surfaces where every simple bundle is stable. For the case $c_2 = 0$ see the remark at the end of this section.

Proof. Let $Z \subset F_q$ be a set of simple points of length c_2. From Proposition 1.1 we get an extension

$$0 \to \mathcal{O}(F_p - F_q) \to E \to J_Z \otimes \mathcal{O}(F_q - F_p) \to 0$$

defining a 2-bundle E with Chern classes $c_1 = 0$, c_2, which is not ω-(semi-)stable because

$$\deg_\omega(\mathcal{O}(F_p - F_q)) = \left(\frac{1}{p} - \frac{1}{q}\right) \deg_{\omega_0}(F) \geqq 0 \quad (>0).$$

That E is simple follows from

$$h^0(\mathcal{O}(2F_p - 2F_q)) = h^0(\mathcal{O}(2F_q - 2F_p)) = 0$$

and Proposition 1.2.

Theorem 1.6. *If $p < q$, then for all $c_2 \geqq 1$ there exists an ω-stable 2-bundle E on X with Chern classes $c_1 = 0$, c_2; if $p \leqq q$, then for all $c_2 \geqq 2$ such a bundle exists.*

Remark. In the case $p = q = m = 2$ (e.g. Enriques surface) there is in fact no stable 2-bundle with $c_1 = 0$, $c_2 = 1$.

Proof. Let $Z \subset F_p$ be a set of simple points of length c_2, and consider the extension

$$0 \to \mathcal{O}(-F_q) \to E \to J_Z \otimes \mathcal{O}(F_q) \to 0$$

which by Proposition 1.1 defines a 2-bundle E with Chern classes $c_1 = 0$, c_2. If E were not stable, from Proposition 1.3 and Corollary 1.4 we would find a vertical curve R containing Z with

$$\deg_\omega(\mathcal{O}(R)) \leqq \deg_\omega(\mathcal{O}(F_q)) = \frac{1}{q} \deg_{\omega_0}(\mathcal{O}(F)).$$

On the other hand, since all fibres are irreducible we can write $R \sim aF + \sum_{i=1}^{m} x_i F_{p_i}$ with $0 \leqq a$, $0 \leqq x_i < p_i$, to get

$$\deg_\omega(\mathcal{O}(R)) = \left(a + \sum_{i=1}^{m} \frac{x_i}{p_i}\right) \deg_{\omega_0}(\mathcal{O}(F)).$$

This implies $a=0$, and in the case $p<q$ furthermore $x_1=0$, contradicting $Z \subset R$. In the case $p \leqq q$, $c_2 \geqq 2$ the same argument works if $Z \subset \bigcup_{i=1}^{m} F_{p_i}$ but not contained in a single fibre.

The next two Propositions show, that every ω-semi-stable and on non-algebraic X every simple 2-bundle with $c_1=0$, $1 \leqq c_2 \leqq 1+2p_g$ occurs as an extension (*).

Proposition 1. 7. *Let E be an ω-semi-stable rank-2 bundle over X with $c_1=0$, $1 \leqq c_2 \leqq 1+2p_g$. Then either there is an extension*

$$0 \to \mathcal{O} \to E \to J_Z \to 0$$

with zero-dimensional Z of length c_2, or there exists a line bundle $\mathcal{O}(D)$, $D \sim aF + \sum_{i=1}^{m} x_i F_{p_i}$ with $0 \leqq a$, $0 \leqq x_i < p_i$ (i.e. D is written in "normalized" form) and an extension

$$(*) \qquad 0 \to \mathcal{O}(D-K) \to E \to J_Z \otimes \mathcal{O}(K-D) \to 0$$

where again Z is zero-dimensional of length c_2. In this case one has $0 \leqq a \leqq p_g + m - 2$, so there is an $\mathcal{O}(D)$ maximal with respect to the ω-degree; if the p_i are pairwise relatively prime, the maximal $\mathcal{O}(D)$ is unique and satisfies $h^0(\mathcal{O}(2D-2K))=0$. For $c_2=1$, $Z=\{z\}$ is a simple point in the fixed locus of the linear system $|3K-2D|$; if furthermore the p_i are pairwise relatively prime, the maximal $\mathcal{O}(D)$ and the accompanying point z are uniquely determined by E and satisfy $h^0(J_z \otimes \mathcal{O}(2K-2D))=0$, $a \geqq p_g - 1$.

Proof. The Riemann-Roch formula gives

$$\chi(E) = -c_2 + 2 + 2p_g \geqq 1,$$

so either E or $E(K)$ has a nontrivial section. In the first case, a 1-dimensional part in the zero set of the section would violate the semi-stability of E, hence we get the first extension. In the second case, take a section $s \neq 0$ in $E(K)$ to get an exact sequence

$$0 \to \mathcal{O}(D-K) \to E \to J_Z \otimes \mathcal{O}(K-D) \to 0$$

where D is the 1-dimensional part of the zero-set of s, so Z is zero-dimensional of length c_2.

Since E is semi-stable we have

$$0 \leqq \deg_\omega(\mathcal{O}(D-K)) = \deg_{\omega_0}(D) + nD \cdot F - \deg_{\omega_0}(K)$$

and because D is a curve we get $D \cdot F = 0$ by our choice of n. Writing $D \sim aF + \sum_{i=1}^{m} x_i F_{p_i}$ (normalized), $\deg_\omega(\mathcal{O}(D)) \leqq \deg_\omega(\mathcal{O}(K))$ reads

$$a + \sum_{i=1}^{m} \frac{x_i}{p_i} \leqq p_g - 1 + \sum_{i=1}^{m} \frac{p_i - 1}{p_i}$$

or

$$(**) \qquad a + \sum_{i=1}^{m} \frac{x_i + 1}{p_i} \leqq p_g - 1 + m$$

giving the estimate for a. In particular, the degrees of possible $\mathcal{O}(D)$ are bounded, so there is a maximal one. If the p_i are pairwise relatively prime, $\mathcal{O}(D)$ is determined by its degree and the maximal one is unique. Now assume $h^0(\mathcal{O}(2D - 2K)) \neq 0$. Then we have

$$\deg(\mathcal{O}(D - K)) = \frac{1}{2} \deg(\mathcal{O}(2D - 2K)) \geqq 0,$$

which together with $\deg(\mathcal{O}(D - K)) \leqq 0$ gives equality in (**), implying $\mathcal{O}(D) = \mathcal{O}(K)$ and thus leading back to the case $h^0(E) \neq 0$. At last, assume $c_2 = 1$ and $h^0(J_z \otimes \mathcal{O}(2K - 2D)) \neq 0$. Mainly from the fact that $2K - 2D$ is "even" one obtains $r \in \{p_i\}$ with $h^0(J_z \otimes \mathcal{O}(2K - 2D - F_r)) \neq 0$. Now we tensor (*) with $\mathcal{O}(K - D - F_r)$ to get

$$0 \to \mathcal{O}(-F_r) \to E(K - D - F_r) \to J_z \otimes \mathcal{O}(2K - 2D - F_r) \to 0,$$

and from $h^0(-F_r) = h^1(-F_r) = 0$ we get $h^0(E(K - D - F_r)) \neq 0$, which violates the maximality of $\mathcal{O}(D)$. Since the line bundle associated to

$$2K - 2D \sim (2p_g - 2 - 2a)F + \sum_{i=1}^{m} (2p_i - 2 - 2x_i)F_{p_i}$$

would have a section vanishing in z if we had $2p_g - 2 - 2a > 0$, we get the lower bound for a.

For simple bundles we now consider a non-algebraic surface X, i.e. every curve C in X is vertical: If C were not vertical, i.e. $C \cdot F > 0$, then $(C + kF)^2 > 0$ for $k \gg 0$, hence X is algebraic [2].

Proposition 1.8. *Let E be a simple rank-2 bundle over a non-algebraic X with $c_1 = 0$, $1 \leqq c_2 \leqq 1 + 2p_g$. Then the same conclusion holds as in the second case of Proposition 1.7.*

Proof. The simplicity of E implies $h^0(E)=0$, so again Riemann-Roch gives $h^0(E(K))\neq 0$, leading to the same extension as above; D has to be vertical because X is non-algebraic. The same argument as in the proof of Proposition 1.2 shows $h^0(\mathcal{O}(2D-2K))=0$. Writing $D \sim aF + \sum_{i=1}^{m} x_i F_{p_i}$ (normalized), we have

$$2D-2K \sim 2\left((a-p_g-m+1)F + \sum_{i=1}^{m} (x_i+1)F_{p_i}\right),$$

so $h^0(\mathcal{O}(2D-2K))=0$ implies $a \leqq p_g+m-2$. The rest of the proof is as for Proposition 1.7.

Remark. For $m=1, 2$ on non-algebraic X there is no simple 2-bundle with $c_1=0$, $c_2=0$, because it would have a maximal divisor satisfying

$$h^0(2D-2K)=h^0(2K-2D)=0 \quad \text{and} \quad h^0(3K-2D)>p_g+1$$

which is impossible. This had to be expected for the following reason: If E is an irreducible rank-2 Hermitian-Einstein bundle (see the introduction; such a bundle is ω-stable and therefore simple) on X with $c_1=0$, $c_2=0$, it is induced by an irreducible unitary representation of $\pi_1(X)$ of rank 2 [10], but for $m=2$ our X has abelian fundamental group (see Section 2), so every irreducible unitary representation has rank 1.

2. Moduli of bundles over simply connected elliptic surfaces

Let X be a minimal elliptic surface over $\mathbb{P}_1$ with multiple fibres $F_{p_1}, \dots, F_{p_m}$ of multiplicities $p_1, \dots, p_m$. The fundamental group of X is abelian if and only if m is at most 2; in this case the order of this group is g.c.d.(p_1, p_2) [3]. Since multiple fibres can be removed using logarithmic transformations, we have the following characterization of simply connected minimal elliptic surfaces.

Proposition 2.1. *The simply connected minimal elliptic surfaces over $\mathbb{P}_1$ are precisely those surfaces which can be obtained by at most two logarithmic transformations with relatively prime multiplicities $p, q \geqq 1$ from a simply connected minimal elliptic surface without multiple fibres.*

In the sequel $X_{p,q}$ will always denote an elliptic surface obtained by at most two logarithmic transformations of relatively prime multiplicities p, q from an elliptic surface X over $\mathbb{P}_1$ without multiple fibres. This surface X is either rational $(p_g(X)=0)$, i.e. a Halphen pencil, or an elliptic $K3$ surface $(p_g(X)=1)$ or properly elliptic. This follows immediately from the canonical bundle formula and the classification of compact surfaces [2].

$X_{p,q}$ is not necessarily algebraic, even if X was algebraic. In fact, $X_{p,q}$ is algebraic if and only if it contains horizontal curves. On the other hand, by the work of Miyaoka [17], an elliptic surface has a Kähler structure if and only if its first Betti number is even. So we can always find a Kähler class on $X_{p,q}$.

Proposition 2. 2. *For all $p_g \geqq 0$ there exist surfaces $X_{p,q}$ without reducible fibres.*

Proof. For any $d \geqq 1$ and any section s of $\mathcal{O}_{\mathbb{P}_1}(d) \times \mathcal{O}_{\mathbb{P}_2}(3)$ over $\mathbb{P}_1 \times \mathbb{P}_2$ take $X = (s)_0$ and let $\pi: X \longrightarrow \mathbb{P}_1$ denote the induced projection. For generic s, X will be smooth with irreducible fibres. From $K_X = \pi^*(\mathcal{O}_{\mathbb{P}_1}(d-2))$ we see $p_g(X) = d-1$.

Remark. Using Freedman's results, it is easy to determine the topology of the surfaces $X_{p,q}$ [8], [15]. The homeomorphism type of $X_{p,q}$ is determined by the geometric genus p_g alone if p_g is even; it is determined by p_g and the parity of $p+q$ if p_g is odd. More precisely, let X_0 denote a Kummer surface and $\bar{\mathbb{P}}_2$ the projective plane with orientation reserved. Then $X_{p,q}$ is homeomorphic to the connected sum

$$\Big(\mathop{\#}_{\frac{1}{2}(p_g+1)} X_0\Big) \# \Big(\mathop{\#}_{\frac{1}{2}(p_g-1)} S^2 \times S^2\Big)$$

if p_g is odd and $p+q$ even; in all other cases is $X_{p,q}$ homeomorphic to

$$\Big(\mathop{\#}_{2p_g+1} \mathbb{P}_2\Big) \# \Big(\mathop{\#}_{10p_g+9} \bar{\mathbb{P}}_2\Big).$$

The surfaces $X_{p,q}$ with $p_g = 1$ and $p+q$ even are the so called homotopy-$K3$-surfaces, first studied by Kodaira [12].

For the rest of this section we denote by $X = X_{p,q}$ an elliptic surface over $\mathbb{P}_1$ with $p_g \geqq 1$, without reducible fibres and with at most two multiple fibres F_p, F_q of relatively prime multiplicities $p, q \geqq 1$ (see Proposition 2. 2), so we can use the results of Section 1. In particular, we fix a Kähler form ω on X as in the remark before Corollary 1. 4.

The first step to classify all semi-stable (simple) rank-2 bundles with $c_1 = 0$, $c_2 = 1$ on an algebraic (non-algebraic) X is according to Propositions 1. 7, 1. 8 the following:

To find possible maximal vertical divisors defining semi-stable or simple bundles, determine all pairs (D, z) where $D \sim aF + bF_p + cF_q$ (normalized) is a vertical curve with $p_g - 1 \leqq a \leqq p_g$, $h^0(\mathcal{O}(2D - 2K)) = 0$, and z is a simple point in the fixed locus of the linear system $|3K - 2D|$ such that $h^0(J_z \otimes \mathcal{O}(2K - 2D)) = 0$.

In principle this can be done for all pairs (p, q), but we restrict ourselves to the cases $p = 1, 2, 3$.

Proposition 2. 3. *For $p = 1, 2, 3$ the pairs (D, z) with the properties just described are as follows:*

i) *For $p = 1$ no such curve exists.*

ii) *For $p = 2$ there are two different types:*

I) $D \sim (p_g - 1)F + F_2 + cF_q$, $z \in F_2$, $\dfrac{q-1}{2} \leqq c \leqq q-2$,

II) $D \sim p_g F + cF_q$, $z \in F_2$, $0 \leqq c \leqq \dfrac{q-3}{2}$.

iii) *For* $p=3$ *there are five types*:

I) $D \sim (p_g-1)F + F_3 + (q-1)F_q,\ z \in F_q,$

II) $D \sim (p_g-1)F + 2F_3 + cF_q,\ z \in F_3,\ \frac{q-1}{2} \leqq c \leqq q-2,$

III) $D \sim p_g F + cF_q,\ z \in F_q,\ \frac{q-2}{2} \leqq c \leqq q-2,$

IV) $D \sim p_g F + F_3 + cF_q,\ z \in F_q,\ c \leqq \frac{q-4}{2},$

V) $D \sim p_g F + F_3 + cF_q,\ z \in F_3,\ c \leqq \frac{q-3}{2}.$

Proof. Tedious computation.

Proposition 2. 4. *The classes of maximal divisors are*:

i) $\tilde{\mathrm{I}}$) $D \sim (p_g-1)F + F_2 + cF_q,\ z \in F_2,\ \frac{3q-3}{4} \leqq c \leqq q-2,$

$\tilde{\mathrm{II}}$) $D \sim p_g F + cF_q,\ z \in F_2,\ \frac{q-3}{4} \leqq c \leqq \frac{q-3}{2}$

for $p=2$.

ii) $\tilde{\mathrm{II}}$) $D \sim (p_g-1)F + 2F_3 + cF_q,\ z \in F_3,\ \frac{5q-5}{6} \leqq c \leqq q-2,$

$\tilde{\mathrm{III}}$) $D \sim p_g F + cF_q,\ z \in F_q,\ \frac{2q-4}{3} \leqq c \leqq q-2,$

$\tilde{\mathrm{IV}}$) $D \sim p_g F + F_3 + cF_q,\ z \in F_q,\ \frac{q-4}{3} \leqq c \leqq \frac{q-4}{2},$

$\tilde{\mathrm{V}}$) $D \sim p_g F + F_3 + cF_q,\ z \in F_3,\ \frac{q-5}{6} \leqq c \leqq \frac{q-3}{2}$

for $p=3$.

Proof. i) Let D, D' be as in Proposition 2. 3 ii) with $\deg(D) < \deg(D')$ and E a bundle given by

$$(*) \qquad 0 \to \mathcal{O}(D-K) \to E \to J_z \otimes \mathcal{O}(K-D) \to 0.$$

Tensor $(*)$ with $\mathcal{O}(K-D')$ to get

$$0 \to \mathcal{O}(D-D') \to E(K-D') \to J_z \otimes \mathcal{O}(2K-D-D') \to 0.$$

Of course $\deg(D) < \deg(D')$ implies $h^0(\mathcal{O}(D-D')) = 0$. If D and D' are of the same type, one sees

$$h^0(J_z \otimes \mathcal{O}(2K-D-D')) = 0,$$

$E(K-D')$ has no section. If D and D' are of different types, it is easy to check that $h^0(\mathcal{O}(K+D'-D))=p_g+1$, so $h^1(\mathcal{O}(D-D'))=0$ by Riemann-Roch and

$$h^0(E(K-D'))=h^0(J_z\otimes\mathcal{O}(2K-D-D')).$$

It follows that D is maximal if and only if $h^0(J_z\otimes\mathcal{O}(2K-D-D'))=0$ for all D' with $\deg(D)<\deg(D')$. Using this criterion the proof is by computation.

ii) Eliminate all (D, z) in Proposition 2.3 iii) for which there exists a D' with $\deg(D)<\deg(D')$, $h^1(\mathcal{O}(D-D'))=0$ and $h^0(J_z\otimes\mathcal{O}(2K-D-D'))\neq 0$. To see that the remaining pairs (D, z) are all maximal it suffices to check $h^0(J_z\otimes\mathcal{O}(2K-D-D'))=0$ for every other remaining D' with $\deg(D)<\deg(D')$.

Corollary 2.5. *For $p=2, 3$ and non-algebraic X the isomorphism classes of simple rank-2 bundles on X with $c_1=0$, $c_2=1$, are in one to one correspondence with pairs $(\mathcal{O}(D), z)$ where (D, z) are as in Proposition 2.4. The ω-stable bundles over algebraic or non-algebraic X are exactly those defined by (D, z) satisfying* $\deg(D-K)<0$. *The ω-semi-stable bundles which are not stable are exactly those defined by an extension*

$$0\longrightarrow\mathcal{O}\longrightarrow E\longrightarrow J_z\longrightarrow 0$$

where z is a simple point in $F_p \amalg F_q$.

Proof. Every simple bundle defines a unique maximal pair $(\mathcal{O}(D), z)$ by Proposition 1.8. Conversely, any (D, z) in our list satisfies $h^1(\mathcal{O}(2D-2K))=0$, so by Proposition 1.1 it defines a unique bundle E. Since all divisors on the list satisfy the conditions of Proposition 1.2, E is simple.

From Proposition 1.3 and using the numerical criterion in Corollary 1.4 it is easy to check that the ω-stable bundles are exactly those with $\deg(D-K)<0$, and that the remaining D define bundles which are not even ω-semi-stable. Thus the last statement follows from Proposition 1.7.

So far we have determined the moduli spaces as sets; now we are going to look at their algebraic (analytic) structures.

First let us consider the case where X is algebraic. We take an ample line bundle L_X on X, then for $n\gg 0$ the line bundle $L=L_X\otimes\mathcal{O}(nK)$ is again ample, and the Kähler form ω induced by L has the properties we need to use the results of Section 1. Denote by $M_{p,q}$ the moduli space of L-stable rank-2 bundles on $X_{p,q}$ with Chern classes $c_1=0$, $c_2=1$. For every pair $r_1\leqq r_2$ of rational numbers let $[r_1, r_2]$ denote the number of integers m with $r_1\leqq m\leqq r_2$.

Theorem 2.6. i) $M_{1,q}=\emptyset$.

ii) $M_{2,q}=\coprod_\mu F_2$ *with* $\mu=\left[\frac{3q-3}{4}, q-2\right]+\left[\frac{q-3}{4}, \frac{q-3}{2}\right]$, *in particular,* $M_{2,q}$ *is reduced.*

iii) $(M_{3,q})_{\mathrm{red}}=(\coprod_\nu F_3)\amalg F_q$ *with* $\nu=\left[\frac{5q-5}{6}, q-2\right]+\left[\frac{q-5}{6}, \frac{q-4}{3}\right]$.

Proof. i) There exists no stable bundle on $X_{1,q}$ by Proposition 2. 3.

ii) For $p=2$ it is easy to see that every divisor in Proposition 2. 4 i) defines a stable bundle, so there is an obvious one to one correspondence between pairs $(\mathcal{O}(D), z)$ and points in $\coprod_{\mu} F_2$. It remains to show that this is the moduli space in the scheme-theoretical sense.

Consider an arbitrary component F_2 of $\coprod_{\mu} F_2$ and let D be the corresponding divisor. If Γ denotes the graph of the natural inclusion $F_2 \subset X$, we have the following commutative diagram:

$$\begin{tikzcd}
\Gamma \arrow[r, hook] \arrow[dr, "\cong"'] & F_2 \times X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \\
& F_2 &
\end{tikzcd}$$

Let $L_D := \mathbf{Ext}^1_{\pi_1}(J_\Gamma \otimes \pi_2^*(\mathcal{O}(K-D)), \pi_2^*(\mathcal{O}(D-K)))$, then from standard arguments [21] we find an extension

$$0 \to \pi_2^*(\mathcal{O}(D-K)) \to E \to J_\Gamma \otimes \pi_2^*(\mathcal{O}(K-D)) \otimes \pi_1^*(L_D) \to 0$$

on $F_2 \times X$ where E is locally free, and for every $z \in F_2$ this E restricts on the π_1-fibre $\{z\} \times X$ to the bundle determined by (D, z).

From the universal property of the moduli scheme $M_{2,q}$ we get a bijective morphism

$$\coprod_{\mu} F_2 \to (M_{2,q})_{\mathrm{red}}.$$

To see that $M_{2,q}$ is reduced it suffices to show $h^1(\operatorname{End} E) \leqq 1$ for every L-stable bundle E. For this consider the sequence

$$0 \to E(D) \to \operatorname{End} E \otimes \mathcal{O}(K) \to J_z \otimes E(2K-D) \to 0.$$

By the Riemann-Roch formula we have $\chi(\operatorname{End} E) = 4p_g$, so we have to show $h^0(\operatorname{End} E \otimes \mathcal{O}(K)) \leqq 4p_g$. Since $h^0(E(D)) = 2p_g$ for all D in Proposition 2. 4 i) we have to prove $h^0(J_z \otimes E(2K-D)) \leqq 2p_g$. But $h^0(E(2K-D)) = 2p_g + 1$, so we need a section in $E(2K-D)$ which does not vanish in z.

The restriction of $E(2K-D)$ to F_2 splits:

$$E(2K-D)|_{F_2} = (\mathcal{O}_{F_2}(z) \oplus \mathcal{O}_{F_2}(-z)) \otimes N^*_{F_2/X}$$

where $N^*_{F_2/X}$ is the non-trivial conormal bundle of F_2 in X. Thus there is a unique section of $E(2K-D)|_{F_2}$ not vanishing in z. This section lifts to $E(2K-D)$ if and only if $h^0(E(2K-D-F_2)) \leqq 2p_g$ which is equivalent to $h^0(J_z(3K-2D-F_2)) \leqq p_g$. This can be verified by calculation.

iii) The restrictions on the values c in Proposition 2.4 ii) coming from $\deg(D)<\deg(K)$ are for the different types as follows:

$\widetilde{\mathrm{II}}$) $\frac{5q-5}{6}\leqq c\leqq q-2$,

$\widetilde{\mathrm{III}}$) $c=\frac{2q-4}{3}$ (so $q=2\ (\mathrm{mod}\ 3)$),

$\widetilde{\mathrm{IV}}$) $c=\frac{q-4}{3}$ (so $q=1\ (\mathrm{mod}\ 3)$),

$\widetilde{\mathrm{V}}$) $\frac{q-5}{6}\leqq c\leqq\frac{q-4}{3}$.

This gives $M_{3,q}$ as a set; the construction of the universal family E is as above. But using the same kind of arguments as for $p=2$ one sees that almost all components of $M_{3,q}$ are non-reduced.

Now let X be non-algebraic and denote by ${}^sM_{p,q}$ the analytic moduli space of simple rank-2 bundles on X with $c_1=0$, $c_2=1$.

Theorem 2.7. i) ${}^sM_{1,q}=\emptyset$.

ii) ${}^sM_{2,q}=\coprod_\mu F_2$ *with* $\mu=\left[\frac{3q-3}{4},q-2\right]+\left[\frac{q-3}{4},\frac{q-3}{2}\right]$.

iii) $({}^sM_{3,q})_{\mathrm{red}}=(\coprod_{\lambda_1} F_3)\amalg(\coprod_{\lambda_2} F_q)$ *with*

$$\lambda_1=\left[\frac{5q-5}{6},q-2\right]+\left[\frac{q-5}{6},\frac{q-3}{2}\right],$$

$$\lambda_2=\left[\frac{2q-4}{3},q-2\right]+\left[\frac{q-4}{3},\frac{q-4}{2}\right].$$

Proof. Set-theoretically this is again obtained from Proposition 2.4 and Corollary 2.5; the construction of the universal family and the proof of the reducedness is as above.

We close with a final

Remark. We have ${}^\omega M_{p,q}={}^sM_{p,q}$ for $p=1,2$, whereas there are many components in ${}^sM_{p,q}\setminus{}^\omega M_{p,q}$ for $p=3$. The description of ${}^\omega M_{p,q}$ is the same as of $M_{p,q}$ for algebraic X, even if ω is not induced by an ample line bundle. Moreover, the moduli spaces are independent of the Kähler form ω_0 (resp. the ample line bundle L_X) we started with to get ω (resp. L). In the algebraic case there may exist simple bundles which are not defined by vertical divisors.

References

[1] *C. Bănică, M. Putinar, G. Schumacher*, Variation der globalen Ext in Deformationen kompakter komplexer Räume, Math. Ann. **250** (1980), 135—155.
[2] *W. Barth, C. Peters, A. van de Ven*, Compact complex surfaces, Erg. Math. **4**, Berlin-Heidelberg-New York 1984.
[3] *I. Dolgachev*, Algebraic surfaces with $q = p_g = 0$, C.I.M.E. (1981).
[4] *S. K. Donaldson*, Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. **50** (1985), 1—26.
[5] *S. K. Donaldson*, The differential topology of complex surfaces, C.R.A.S. **301**, (1) n° 6 (1985), 317.
[6] *S. K. Donaldson*, Connections, cohomology and the intersection forms of 4-manifolds, J. Diff. Geom. **24** (1986), 275—341.
[7] *D. S. Freed, K. K. Uhlenbeck*, Instantons and four-manifolds, M.S.R.I. Publ. **1**, Berlin-Heidelberg-New York 1984.
[8] *M. Freedman*, The topology of four dimensional manifolds, J. Diff. Geom. **17** (1982), 357—453.
[9] *S. Kobayashi*, Curvature and stability of vector bundles, Proc. Japan Acad. **58** A4 (1982), 158—162.
[10] *S. Kobayashi*, Differential geometry of holomorphic vector bundles, PAM **315**, Berkeley 1986.
[11] *K. Kodaira*, On the structure of compact complex analytic surfaces. I, Am. J. Math. **86** (1964), 751—798.
[12] *K. Kodaira*, On homotopy-$K3$ surfaces. Essays on topology and related topics (ded. G. de Rham), Berlin-Heidelberg-New York 1970.
[13] *M. Lübke*, Stability of Einstein-Hermitian vector bundles, manuscripta math. **42** (1983), 245—257.
[14] *M. Lübke, C. Okonek*, Moduli spaces of simple bundles and Hermitian-Einstein connections, Math. Ann. **276** (1987), 663—674.
[15] *R. Mandelbaum*, On the topology of elliptic surfaces, Adv. math. suppl. studies **3** (1979).
[16] *M. Maruyama*, Moduli of stable sheaves. II, J. Math. Kyoto Univ. **18** (1978), 557—614.
[17] *Y. Miyaoka*, Kähler metrics on elliptic surfaces, Proc. Japan Acad. **50** (1974), 533—536.
[18] *S. Mukai*, Symplectic structure of the moduli space of sheaves on an Abelian or $K3$ surface, Invent. math. **77** (1984), 101—116.
[19] *A. Norton*, Analytic moduli of complex vector bundles, Indiana Univ. Math. J. **28** (1979), 365—387.
[20] *C. Okonek, M. Schneider, H. Spindler*, Vector bundles on complex projective spaces, Progress math. **3**, Boston 1980.
[21] *C. Okonek, A. van de Ven*, Stable bundles and differentiable structures on certain elliptic surfaces, Invent. math. **86** (1986), 357—370.
[22] *C. H. Taubes*, Self-dual connections on 4-manifolds with indefinite intersection matrix, J. Diff. Geom. **19** (1984), 517—560.
[23] *K. Uhlenbeck, S. T. Yau*, On the existence of Hermitian-Yang-Mills connections in stable vector bundles, preprint 1986.

Mathematisch Instituut, Rijksuniversiteit Leiden, Niels Bohrweg 1, NL-2300 RA Leiden

SFB 170, Mathematisches Institut, Bunsenstraße 3–5, D-3400 Göttingen

Eingegangen 16. Juni 1986